Generating continuous RVs

Wednesday, February 26, 2014 8:46 AM

Feb 26, 2013

Announcements

1. HW 6 ... Due tomorrow ← can still ask question.
2. No HW 7 ... replaced by in-class exercise.
3. Second project ... Due next week (last day of class). (Mar 8)

Objectives of this class

1. An opportunity for students to practice using MATLAB
2. Introduce the important ideas in Monte Carlo simulation

Big picture of the remaining topics:

Generation of RVs: Discrete / Continuous

Discrete Event Simulation ← Generation of HPP and NHPP

Monte Carlo numerical algorithms

In this class, there are multiple ways that I use to check your understanding of the presented material:

1) Questions raised in class (from you and me)
   - Best because we can talk about them immediately.
   - This is why I often ask questions and also encourage you to ask questions in class.

2) HW
   - I look through your HWs to see which part(s) of the presented lecture(s) need to be reviewed.
   - This is why we have several HWs.

3) In-class exercise
   - Encourage discussion in class

Review

There are more techniques for generating RVs than we can cover in class. References are provided in the slides.

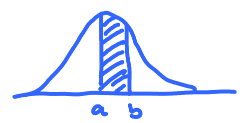
Last time : discrete RVs and their generation

Today : continuous " "

Generating Continuous RVs

Some transition that we need to make when we consider continuous RVs

Discrete	Continuous
pmf $p_X(x)$ ↑ mass	pdf $f_X(x)$ ↑ density
	$P[a < X < b] = \int_a^b f_X(x)\,dx$
cdf: $F_X(x) = P[X \le x]$	$F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt$ $\frac{d}{dx} F_X(x) = f_X(x)$



Fundamental theorem of calculus

Technique #1: Inverse transform method

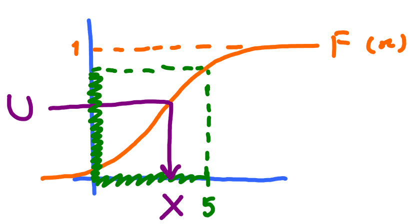
$$X = F_X^{-1}(U) \quad \text{where} \quad U \sim \mathcal{U}(0,1)$$

Rough idea on why inverse transform method works:

Consider any function $F(x)$ that satisfies the three characterizing properties of cdf.

Let $X = F^{-1}(U)$ where $U \sim \mathcal{U}(0,1)$.

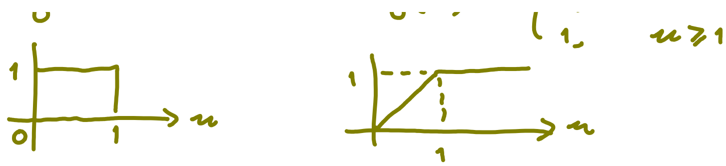
Let's try to find $F_X(x)$.



$$F_X(5) = P[X \le 5] = P[U \le \overset{u}{F(5)}]$$
$$= F_U(F(5))$$
$$= F(5)$$

$f_U(u)$

$$F_U(u) = \begin{cases} 0, & u \le 0 \\ u, & 0 < u < 1 \\ 1, & u \ge 1 \end{cases}$$

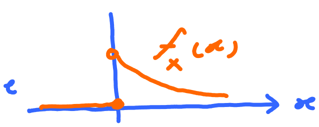


Similarly, $F_X(x) = F(x)$ for any x.
So, by setting $X = F^{-1}(U)$, we can generate X with arbitrary cdf F.

Ex. Suppose we want to generate an exponential RV with parameter λ. ($X \sim \mathcal{E}(\lambda)$.)

First, recall that

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & x > 0 \\ 0, & \text{otherwise} \end{cases}$$



$$F_X(x) = \begin{cases} \int_0^x f_X(t)\, dt, & x > 0 \\ 0, & x \le 0 \end{cases} \underset{\text{Integration by MATLAB}}{=} \begin{cases} 1 - e^{-\lambda x}, & x > 0 \\ 0, & x \le 0 \end{cases}$$

Now that we have $F_X(x)$. The next step is to find its inverse.

$$y = 1 - e^{-\lambda x}$$

swap the x and y

$$x = 1 - e^{-\lambda y}$$

$$e^{-\lambda y} = 1 - x$$

$$-\lambda y = \ln(1-x)$$

$$y = -\frac{1}{\lambda}\ln(1-x)$$

Therefore, we can set

$$X = -\frac{1}{\lambda}\ln(1-U)$$

$1-U$ can be replaced by U because they have the same pdf.

Conclusion: To generate $X \sim \mathcal{E}(\lambda)$, set $X = -\frac{1}{\lambda}\log U$ where $U \sim \mathcal{U}(0,1)$.

To verify the resulting RV, we may check its empirical cdf or pdf.
For example, you can overlay a theoretical cdf on the same plot as the empirical one to compare them.

(i) Empirical cdf:

Suppose we want to evaluate $F_X(5)$ from our data. generated by

(In our example above, theoretically $F_X(5) = 1 - e^{-5\lambda}$.)

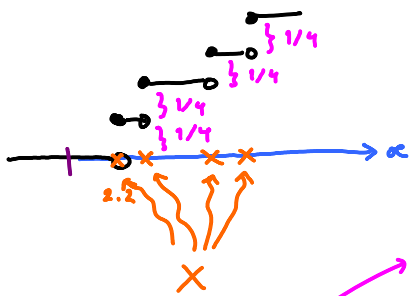
By definition, $F_X(5) = P[X \leq 5]$.

This can be estimated from the data by

$$F_X(5) = P[X \leq 5] \approx \frac{\# X\text{'s} \leq 5}{\# X\text{'s}}.$$

In general, the empirical cdf at x is then simply the proportion of X values less than or equal to x.

We try an example with four X values.

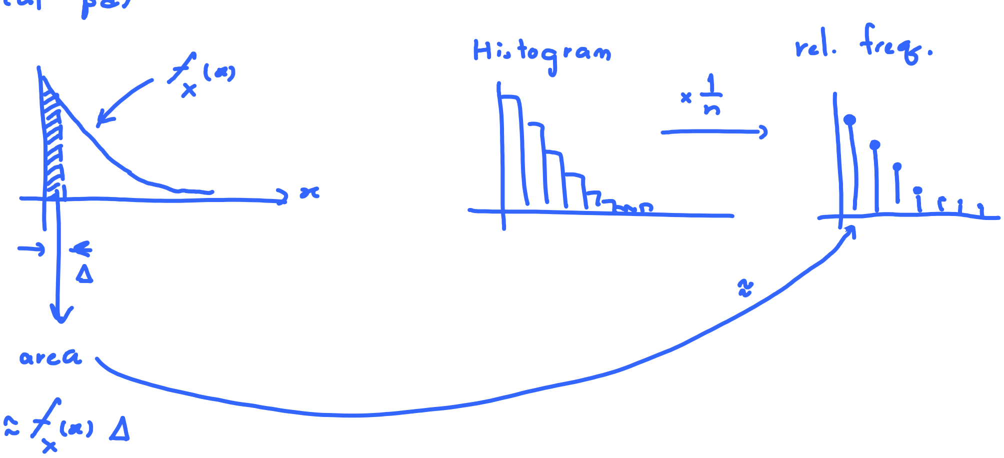


Note that

1) It is a staircase function (similar to what we saw for discrete RV.)
2) Jumps occur at the X values
3) The jump sizes are all $\frac{1}{n}$ where n is the total number of X values.

These observations make it easy to code a function to plot the empirical cdf in MATLAB.

(ii) Empirical pdf



conclusion: We can get the empirical pdf via scaling the histogram by $\frac{1}{n\Delta}$.

Technique #2: Acceptance-Rejection Method

See slides.

Generating cont. RV with infinite support

Infinite area

Generating cont. RV with infinite support

Can't have uniform distributed pts over infinite area.

Area under the graph of any pdf is 1

We can scale the graph vertically by c to cover larger area.

Ex. Suppose we want to generate a RV X whose

$$f_X(x) = \begin{cases} \frac{2}{\sqrt{2\pi}} e^{-x^2/2}, & 0 < x < \infty \\ 0, & \text{otherwise} \end{cases}$$

Recipe: ⓪ Find $Y \sim \mathcal{E}(1)$ whose $f_Y(y)$ is known and generable and support is larger than the support of X

$$f_Y(y) = \begin{cases} e^{-y}, & y > 0 \\ 0, & \text{otherwise} \end{cases}$$

Find c such that $\frac{f_X(t)}{f_Y(t)} \leq c \quad \forall t$

① Generate $Y \sim f_Y$

② Generate $U \sim \mathcal{U}(0,1)$

③ If $U \leq \frac{f_X(Y)}{c f_Y(Y)}$, set $X = Y$

Otherwise, return to step ①

Finding the best c : Note that large value of c means we have to reject many points.
So, we want c to be small.
However, we also need $c \geq \frac{f_X}{f_Y}$. So, we should first find the maximum of f_X / f_Y and set c to be that value.

f_X / f_Y

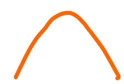
In our example, $\dfrac{f_X(t)}{f_Y(t)} = \dfrac{\frac{2}{\sqrt{2\pi}} e^{-t^2/2}}{e^{-t}} = \dfrac{2}{\sqrt{2\pi}} e^{-\frac{t^2}{2} + t}$

To maximize the ratio,

we want to maximize $-\frac{t^2}{2} + t$

$\frac{d}{dt}$: $-t + 1 = 0$

$t = 1$



$$\max \frac{f_X}{f_Y} = \frac{2}{\sqrt{2\pi}} \sqrt{e} = \sqrt{\frac{2e}{\pi}}$$

"best" $c = \sqrt{\frac{2e}{\pi}}$

The above example was solved using $Y \sim \mathcal{E}(1)$. Let's try $Y \sim \mathcal{E}(\lambda)$.

$$\frac{f_X(t)}{f_Y(t)} = \frac{2}{\sqrt{\pi}} e^{-t^2/2} \Big/ \lambda e^{-\lambda t} = \frac{2}{\sqrt{\pi}\lambda} e^{-t^2/2 + \lambda t}$$

max occurs when $\frac{d}{dt}\left(-t^2/2 + \lambda t\right) = 0$

$$-t + \lambda = 0$$

$$t = \lambda.$$

The corresponding max value is $\frac{2}{\sqrt{\pi}\lambda} e^{-\lambda^2/2 + \lambda^2} = \frac{2}{\sqrt{\pi}\lambda} e^{\lambda^2/2}$.

So, given a λ, the best $c = \frac{2}{\sqrt{\pi}\lambda} e^{\lambda^2/2}$.

Again, we want c to be as small as possible.

So, we try to find the best λ to minimize c.

MATLAB give $\lambda = 1$ which is the case that we consider originally

Exercise : Suppose we want to generate a RV X whose

$$f_X(x) = \begin{cases} x e^{-x}, & x > 0, \\ 0, & \text{otherwise.} \end{cases}$$

We will generate X by using the acceptance-rejection method with $Y \sim \mathcal{E}(\lambda)$.

① (60%) Find λ, c that work.

② (40%) Find the best λ, c (that work).

(show your work)

Solution

(a) $f_Y(y) = \begin{cases} \lambda e^{-\lambda y}, & y>0, \\ 0, & \text{otherwise}. \end{cases}$

$$\frac{f_X(t)}{f_Y(t)} = \frac{te^{-t}}{\lambda e^{-\lambda t}} = \frac{t}{\lambda} \frac{1}{e^{(1-\lambda)t}}$$

Note that if $\lambda \geq 1$, $\frac{f_X(t)}{f_Y(t)} \to \infty$ as $t \to \infty$.

(max $= \infty$)

So, we only consider $0 < \lambda < 1$.

The max value of the ratio $\frac{f_X}{f_Y}$ occurs at $t = \frac{1}{1-\lambda}$

with corresponding max value of $\frac{1}{\lambda(1-\lambda)e}$.

So, given λ, the best c is $\frac{1}{\lambda(1-\lambda)e}$.

Conclusion: choose any $0 < \lambda < 1$ and then

choose any $c \geq \frac{1}{\lambda(1-\lambda)e}$.

(b) We then minimize the above c to get the optimal λ.

The min of $\frac{1}{\lambda(1-\lambda)e}$ occurs at $\lambda = \frac{1}{2}$

with corresponding max value of $\frac{4}{e}$.